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LINEAR SYSTEMS OF PLANE CURVES WITH A COMPOSITE NUMBER OF BASE POINTS OF EQUAL MULTIPLICITY

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ABSTRACT. In this article we study linear systems of plane curves of degree d passing through general base points with the same multiplicity at each of them. These systems are known as *homogeneous* linear systems. We especially investigate for which of these systems, the base points, with their multiplicities, impose independent conditions and which homogeneous systems are empty. Such systems are called *non-special*. We extend the range of homogeneous linear systems that are known to be non-special. A theorem of Evain states that the systems of curves of degree d with 4^h base points with equal multiplicity are non-special. The analogous result for 9^h points was conjectured. Both of these will follow, as corollaries, from the main theorem proved in this paper. Also, the case of 4^h9^k points will follow from our result. The proof uses a degeneration technique developed by C. Ciliberto and R. Miranda.

INTRODUCTION

Fix the projective plane $\mathbb{P}^2$ and n general points $q_1, \ldots, q_n$ on it. Consider the linear system consisting of plane curves of degree d with multiplicity at least m_i at each point q_i. In this paper we will restrict ourselves to linear systems of curves which pass through every point with the same fixed multiplicity m, i.e., $m_i = m$ for all $i = 1, \ldots, n$. Such systems are called *homogeneous* and are denoted by $\mathcal{L}_d(m^n)$.

The linear system of all plane curves of degree d has projective dimension

$$\binom{d+2}{2} - 1 = \frac{d(d+3)}{2}.$$

Each base point of multiplicity m imposes $\frac{m(m+1)}{2}$ conditions. These conditions come from the vanishing of the coefficients in the Taylor expansion of the equation of the curve. Define the *virtual dimension* of $\mathcal{L}_d(m^n)$ by

$$v = v(d, m, n) = \frac{d(d+3)}{2} - n\,\frac{m(m+1)}{2}.$$

The actual dimension of this system cannot be less than -1 and if the dimension is equal to -1, then the system is empty. Therefore we define the *expected dimension*

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to be

$$e = e(d, m, n) = \max\{-1,\ v\}.$$

Since we chose the points $q_1, \ldots, q_n$ in general position, the dimension of $\mathcal{L}_d(m^n)$ achieves its minimum value. We call this the *dimension* of $\mathcal{L}_d(m^n)$ and denote it by l. Observe that $l \geq e$. Equality implies that either the system is empty, or the conditions imposed by the multiple points are independent.

The system $\mathcal{L}_d(m^n)$ is *non-special* if $l = e$. Otherwise, we say that the system is *special.*

The main result of this paper is the following theorem.

Theorem 1. *If all homogeneous linear systems, of all degrees and multiplicities, through n_1 points in general position are non-special and all the homogeneous systems through n_2 points in general position are non-special, then the linear systems $\mathcal{L}_d(m^{n_1 n_2})$ are also non-special, for all d and m.*

We are actually able to prove more. The following is a slightly more precise version of the previous theorem, which actually uses the full strength of our proof.

Theorem 2. *For fixed d, m, n_1, n_2 as above, and for k an integer between*

$$\frac{1}{2}\left(-3+\sqrt{1+4n_2m(m+1)}\right) \quad \text{and} \quad \frac{1}{2}\left(-1+\sqrt{1+4n_2m(m+1)}\right),$$

the linear system $\mathcal{L}_d(m^{n_1 n_2})$ will be non-special if the systems

$$\mathcal{L}_d((k+1)^{n_1}), \quad \mathcal{L}_d(k^{n_1}), \quad \mathcal{L}_{k-1}(m^{n_2}) \quad \text{and} \quad \mathcal{L}_k(m^{n_2})$$

are non-special.

This is a generalization of Evain's theorem [4] in which it is proved that all systems of plane curves of degree d through 4^h points with homogeneous multiplicity m are non-special. We prove our result by using a degeneration technique developed by C. Ciliberto and R. Miranda [1], which we now describe.

1. Degeneration Technique

We will now describe the degeneration technique needed for the proof. We first consider a degeneration of the plane $\mathbb{P}^2$, then a degeneration of the bundle and finally of the base points.

Let us start by degenerating $\mathbb{P}^2$. Let Δ be a complex disc around the origin. The product $V = \mathbb{P}^2 \times \Delta$ is equipped with two projections $p_1 : V \to \Delta$ and $p_2 : V \to \mathbb{P}^2$. We denote $V_t = \mathbb{P}^2 \times \{t\}$.

Consider n_1 general points in the plane V_0 and blow up V at these points. We get a new threefold X with maps $f : X \to V$, $\pi_1 = p_1 \circ f : X \to \Delta$ and $\pi_2 = p_2 \circ f : X \to \mathbb{P}^2$. The map π_1 gives a flat family of surfaces over Δ.

$$\begin{array}{ccc} X & & \\ \big\downarrow{\scriptstyle f} & & \\ V & \xrightarrow{p_2} & \mathbb{P}^2 \\ \big\downarrow{\scriptstyle p_1} & & \\ \Delta & & \end{array}$$

Let X_t be the fiber of π_1 over $t \in \Delta$. If $t \neq 0$, then $X_t \cong V_t$ is a plane $\mathbb{P}^2$. By contrast, X_0 is the union of the proper transform of V_0, which we denote by Y, and of n_1 exceptional divisors $\mathbf{P}_i$ for $i = 1, \dots, n_1$. Each $\mathbf{P}_i$ is isomorphic to the plane $\mathbb{P}^2$. The variety Y is $\mathbb{P}^2$ blown up at n_1 points. Denote this blow-up by $b : Y \to \mathbb{P}^2$. Each $\mathbf{P}_i$ intersects Y transversally along a curve E_i, which is a line in $\mathbf{P}_i$ and an exceptional divisor on Y.

Next we consider a degeneration of the bundle $\mathcal{O}_{\mathbb{P}^2}(d)$. Define a line bundle on X by

$$\mathcal{O}_X(d,k) = \pi_2^*\mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{O}_X(kY).$$

The restriction of $\mathcal{O}_X(d,k)$ to X_t, for $t \neq 0$, is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(d)$.

Let $\mathcal{X}(d,k)$ be the restriction of $\mathcal{O}_X(d,k)$ to X_0. $\mathcal{X}(d,k)$ is a flat limit of the bundle $\mathcal{O}_{\mathbb{P}^2}(d)$ on the general fiber X_t. On each $\mathbf{P}_i$, the bundle $\mathcal{X}(d,k)$ is equal to $\mathcal{O}_X(d,k)|_{\mathbf{P}_i}$ which is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(k)$, since Y intersects $\mathbf{P}_i$ along a line. On the Y bundle, $\mathcal{X}(d,k)$ is the tensor product of

$$\pi_2^*\mathcal{O}_{\mathbb{P}^2}(d)|_Y \cong \mathcal{O}_Y(b^*(dL))$$

and of

$$\mathcal{O}_X(kY)|_Y \cong \mathcal{O}_X(kY - kX_t)|_Y \cong \mathcal{O}_X(-\sum_{i=1}^{n_1} k\mathbf{P}_i)|_Y \cong \mathcal{O}_X(-\sum_{i=1}^{n_1} kE_i).$$

The last statement holds since $X_t \sim Y + \sum_1^{n_1} \mathbf{P}_i$ as divisors on X, and $\mathbf{P}_i$ intersects Y along an exceptional line. So

$$\mathcal{X}(d,k)|_Y \cong \mathcal{O}_Y\left(dL - \sum_{i=1}^{n_1} kE_i\right)$$

where we abuse notation and write L for b^*L on Y.

Now fix a positive integer n_2 and put n_2 general points $p_{i,1}, \dots, p_{i,n_2}$ in each $\mathbf{P}_i$. We can view these points as limits of $n_1 n_2$ general points $p_{1,t}, \dots, p_{n_1 n_2, t}$ in X_t, $t \neq 0$. Consider then the linear system $\mathcal{L}_t$ which is the system $\mathcal{L}_d(m^{n_1 n_2})$ in $X_t \cong \mathbb{P}^2$ based at the points $p_{1,t}, \dots, p_{n_1 n_2, t}$.

We also consider the system $\mathcal{L}_0(d,k,m,n_1n_2)$ on X_0 consisting of divisors in $|\mathcal{X}(d,k)|$ which vanish with multiplicity at least m at $p_{1,1}, \dots, p_{1,n_2}, \dots, p_{n_1,n_2}$. As discussed above, $\mathcal{L}_0(d,k,m,n_1n_2)$ on X_0 can be seen as a flat limit of the system $\mathcal{L}_t = \mathcal{L}_d(m^{n_1 n_2})$.

By upper-semicontinuity, the dimension of $\mathcal{L}_0(d,k,m,n_1n_2)$ is at least equal to the dimension of $\mathcal{L}_d(m^{n_1 n_2})$. Therefore, in order to prove that the dimension of $\mathcal{L}_d(m^{n_1 n_2})$ is equal to the expected dimension, it is enough to prove that, under the assumptions of Theorem 2,

$$\dim \mathcal{L}_0(d,k,m,n_1n_2) = e(d,m,n_1n_2).$$

2. The Transversality of the Restricted Systems

Recall from Section 1 the line bundle $\mathcal{X}(d,k)$ on X_0 with

$$\mathcal{X}(d,k)|_Y \cong \mathcal{O}_Y\left(dL - \sum_{i=1}^{n_1} kE_i\right)$$

and

$$\mathcal{X}(d,k)|_{\mathbf{P}_i} \cong \mathcal{O}_{\mathbb{P}^2}(kL).$$

Recall also that we put n_2 points with the same multiplicity m in each $\mathbf{P}_i$. Global sections of $\mathcal{X}(d,k)|_Y$ correspond to curves in $\mathbb{P}^2$ which pass through each of the n_1 points with multiplicity at least k. This implies that the restriction of the system $\mathcal{L}_0(d,k,m,n_1n_2)$ to Y is a system $\mathcal{L}_Y$ isomorphic to $\mathcal{L}_d(k^{n_1})$. In the same way, the restriction of $\mathcal{L}_0(d,k,m,n_1n_2)$ to $\mathbf{P}_i \cong \mathbb{P}^2$ is a system $\mathcal{L}_{\mathbf{P}_i}$ isomorphic to $\mathcal{L}_k(m^{n_2})$.

The system $\mathcal{L}_0$ is a linear system on a reducible scheme X_0. Its elements consist of $\alpha \in \mathcal{L}_Y$ on Y and of $\beta_i \in \mathcal{L}_{\mathbf{P}_i}$ on each $\mathbf{P}_i$ which restrict to the same divisor on the lines E_i.

In order to compute the dimensions of linear systems, it is easier to use the dimensions of the corresponding vector spaces. We denote by $\mathbf{L}_d(k^{n_1})$ the vector space whose projectivization is $\mathcal{L}_d(k^{n_1})$ and similarly we introduce the vector spaces $\mathbf{L}_k(m^{n_2}), \mathbf{L}_Y, \mathbf{L}_{\mathbf{P}_i}$ and $\mathbf{L}_0$ such that

$$\begin{aligned}
&\mathbb{P}(\mathbf{L}_d(k^{n_1})) = \mathcal{L}_d(k^{n_1}),\\
&\mathbb{P}(\mathbf{L}_k(m^{n_2})) = \mathcal{L}_k(m^{n_2}),\\
&\mathbb{P}(\mathbf{L}_Y) = \mathcal{L}_Y,\\
&\mathbb{P}(\mathbf{L}_{\mathbf{P}_i}) = \mathcal{L}_{\mathbf{P}_i},\\
&\mathbb{P}(\mathbf{L}_0) = \mathcal{L}_0(d,k,m,n_1n_2).
\end{aligned}$$

By restriction to the lines $\bigcup_{i=1}^{n_1} E_i$ we have maps

$$\mathbf{L}_Y \xrightarrow{\rho_Y} \bigoplus_{i=1}^{n_1} H^0(E_i, \mathcal{O}_{E_i}(k)) \text{ and } \mathbf{L}_{\mathbf{P}_i} \xrightarrow{r_i} H^0(E_i, \mathcal{O}_{E_i}(k)).$$

At the level of vector spaces, $\mathbf{L}_0$ is the fibered product of $\mathbf{L}_Y$ and $\bigoplus_{i=1}^{n_1} \mathbf{L}_{\mathbf{P}_i}$ over the vector space of restricted systems on $\bigcup_{i=1}^{n_1} E_i$, which is

$$\bigoplus_{i=1}^{n_1} H^0(E_i, \mathcal{O}_{E_i}(k)).$$

This means

$$\mathbf{L}_0 = \mathbf{L}_Y \times_{\oplus_{i=1}^{n_1} H^0(E_i,\mathcal{O}(k))} \left(\bigoplus_{i=1}^{n_1} \mathbf{L}_{\mathbf{P}_i} \right).$$

The situation is shown in the diagram below:

$$\begin{array}{ccccccccc}
\mathbf{L}_0 & \longrightarrow & \mathbf{L}_{\mathbf{P}_1} & \oplus & \mathbf{L}_{\mathbf{P}_2} & \oplus\cdots\oplus & \mathbf{L}_{\mathbf{P}_{n_1}} \\
\downarrow & & \big\downarrow r_1 & & \big\downarrow r_2 & & \big\downarrow r_{n_1} \\
\mathbf{L}_Y & \xrightarrow{\rho_Y} & H^0(E_1, \mathcal{O}_{E_1}(k)) & \oplus & H^0(E_2, \mathcal{O}_{E_2}(k)) & \oplus\cdots\oplus & H^0(E_{n_1}, \mathcal{O}_{E_{n_1}}(k)).
\end{array}$$

Our task is to calculate the dimension of $\mathbf{L}_0$. By the definition of the fibred product

$$\mathbf{L}_0 = \left\{ (\alpha, \beta) \in \mathbf{L}_Y \times \left(\bigoplus_{i=1}^{n_1} \mathbf{L}_{\mathbf{P}_i} \right) \ : \ \alpha|_{\oplus_{i=1}^{n_1} E_i} = \beta|_{\oplus_{i=1}^{n_1} E_i} \right\}.$$

An element of $\mathbf{L}_0$ is determined by first choosing an element γ in $\mathrm{Im}(\rho_Y) \cap \mathrm{Im}(r_1, \dots, r_{n_1})$ and then taking its inverse images $\alpha \in \mathbf{L}_Y$ and $\beta \in (\bigoplus_{i=1}^{n_1} \mathbf{L}_{\mathbf{P}_i})$. Once we fix γ, the choice of α depends on $\dim\ker(\rho_Y)$ parameters and similarly the choice of β depends on $\dim\ker(r_1, \dots, r_{n_1})$ parameters. From these considerations we get

$$(*) \quad \dim \mathbf{L}_0 = \dim(\mathrm{Im}(\rho_Y) \cap \mathrm{Im}(r_1, \dots, r_{n_1})) + \dim\ker(\rho_Y) + \dim\ker(r_1, \dots, r_{n_1}).$$

As proved in [3], we can apply the Generalized Transversality Lemma to the above diagram of vector spaces. By this lemma we can assume that $\mathrm{Im}(\rho_Y)$ and $\mathrm{Im}(r_1, \ldots, r_{n_1})$ intersect properly. This means that

$$\begin{aligned}&\dim(\mathrm{Im}(\rho_Y) \cap \mathrm{Im}(r_1, \ldots, r_{n_1})) \\ &= \max\{\dim \mathrm{Im}(\rho_Y) + \dim \mathrm{Im}(r_1, \ldots, r_{n_1}) - n_1(k+1),\ 0\}.\end{aligned}$$

We can describe divisors in the kernel of ρ_Y as elements in $\mathbf{L}_Y$ which contain $\bigcup_{i=1}^{n_1} E_i$ as a component. These correspond to curves of degree d in $\mathbb{P}^2$ which pass through each point of the blow-up with multiplicity at least $k+1$. This means $\ker(\rho_Y) \cong \mathbf{L}_d((k+1)^{n_1})$. In the same way, divisors in the kernel of r_i are sections of $\mathbf{L}_{\mathbf{P}_i}$ that contain E_i as a component. These correspond to curves of degree $k-1$ in $\mathbf{P}_i$ and therefore $\ker(r_i) \cong \mathbf{L}_{k-1}(m^{n_2})$. This gives rise to left exact sequences

$$0 \longrightarrow \mathbf{L}_d((k+1)^{n_1}) \longrightarrow \mathbf{L}_Y \xrightarrow{\rho_Y} \bigoplus_{i=1}^{n_1} H^0(\mathcal{O}_{E_i}(k)),$$

$$0 \longrightarrow \mathbf{L}_{k-1}(m^{n_2}) \longrightarrow \mathbf{L}_{\mathbf{P}_i} \xrightarrow{r_i} H^0(\mathcal{O}_{E_i}(k)).$$

Recall $\mathcal{L}_Y \cong \mathcal{L}_d(k^{n_1})$ and $\mathcal{L}_{\mathbf{P}_i} \cong \mathcal{L}_k(m^{n_2})$. From this we can calculate

$$\begin{aligned}&\dim \mathrm{Im}(\rho_Y) = \dim \mathbf{L}_d(k^{n_1}) - \dim \mathbf{L}_d((k+1)^{n_1}),\\ &\dim \mathrm{Im}(r_1, \ldots, r_{n_1}) = n_1(\dim \mathbf{L}_k(m^{n_2}) - \dim \mathbf{L}_{k-1}(m^{n_2})),\\ &\dim \ker(\rho_Y) = \dim \mathbf{L}_d((k+1)^{n_1}),\\ &\dim \ker(r_1, \ldots, r_{n_1}) = n_1 \dim \mathbf{L}_{k-1}(m^{n_2}).\end{aligned}$$

Since, by assumption in our theorem, we know that the dimension of all homogeneous systems through n_1 or n_2 points is equal to the expected dimension, we are able to compute the dimensions of all the kernels and images above, and therefore we are able to compute the dimension of $\mathbf{L}_0$.

3. Dimension of the Fibered Product $\mathbf{L}_0$

In the sequel we assume that n_1 and n_2 are such that linear systems $\mathcal{L}_d(m^{n_1})$ and $\mathcal{L}_d(m^{n_2})$ are non-special for every d and m. By the definition of a non-special system, the corresponding vector spaces have dimension

$$\dim \mathbf{L}_d(m^{n_1}) = \max\left\{\frac{d(d+3)}{2} + 1 - n_1\frac{m(m+1)}{2},\ 0\right\}$$

and

$$\dim \mathbf{L}_d(m^{n_2}) = \max\left\{\frac{d(d+3)}{2} + 1 - n_2\frac{m(m+1)}{2},\ 0\right\}.$$

Our task is to prove that then the linear system $\mathcal{L}_d(m^{n_1 n_2})$ is also non-special. As mentioned in Section 1, the dimension of the vector space $\mathbf{L}_0$ on X_0 by semicontinuity satisfies the following inequality:

$$\dim(\mathbf{L}_0) \geq \dim \mathbf{L}_d(m^{n_1 n_2}).$$

Therefore it is enough to prove

$$\dim \mathbf{L}_0 = \max\left\{\frac{d(d+3)}{2} + 1 - n_1 n_2\frac{m(m+1)}{2},\ 0\right\}.$$

Now we are ready to start the computation.

Claim 3.1. In the computation for the dimension of $\mathbf{L}_0$, given d, k, m, n_1, n_2 as above, we have:

$$\dim \ker(\rho_Y) + \dim \ker(r_1, \ldots, r_{n_1})$$

$$= \dim \mathbf{L}_d((k+1)^{n_1}) + n_1 \dim \mathbf{L}_{k-1}(m^{n_2})$$

$$= \begin{cases} \frac{d(d+3)}{2} + 1 - n_1 n_2 \frac{m(m+1)}{2} - n_1(k+1) & \text{if } \dim \mathbf{L}_d((k+1)^{n_1}) \geq 0, \\ & \quad \dim \mathbf{L}_{k-1}(m^{n_2}) \geq 0; \\ \frac{d(d+3)}{2} + 1 - n_1 \frac{(k+1)(k+2)}{2} & \text{if } \dim \mathbf{L}_d((k+1)^{n_1}) \geq 0, \\ & \quad \dim \mathbf{L}_{k-1}(m^{n_2}) = 0; \\ n_1 \left(\frac{(k-1)(k+2)}{2} + 1 - n_2 \frac{m(m+1)}{2} \right) & \text{if } \dim \mathbf{L}_d((k+1)^{n_1}) = 0, \\ & \quad \dim \mathbf{L}_{k-1}(m^{n_2}) \geq 0; \\ 0 & \text{if } \dim \mathbf{L}_d((k+1)^{n_1}) = 0, \\ & \quad \dim \mathbf{L}_{k-1}(m^{n_2}) = 0. \end{cases}$$

Proof. Suppose $\dim \mathbf{L}_d((k+1)^{n_1}) \geq 0$ and $\dim \mathbf{L}_{k-1}(m^{n_2}) \geq 0$. Then $\dim \mathbf{L}_d((k+1)^{n_1}) + n_1 \dim \mathbf{L}_{k-1}(m^{n_2})$ is equal to

$$\frac{d(d+3)}{2} + 1 - n_1 \frac{(k+1)(k+2)}{2} + n_1 \frac{(k-1)(k+2)}{2} + n_1 - n_1 n_2 \frac{m(m+1)}{2},$$

which simplifies to the given expression. This proves the first case. The other cases follow easily, given the definition of expected dimension, since one or both of the terms will be zero. □

Before we continue observe

$$\dim \mathbf{L}_d(k^{n_1}) \geq \dim \mathbf{L}_d((k+1)^{n_1}) \text{ and } \dim \mathbf{L}_k(m^{n_2}) \geq \dim \mathbf{L}_{k-1}(m^{n_2}),$$

and $\dim \mathbf{L}_d(m^{n_i}) = 0$ if and only if $\frac{d(d+3)}{2} + 1 - n_i \frac{m(m+1)}{2} \leq 0$ for $i = 1, 2$. Bearing this in mind we get

$$\dim(\operatorname{Im}(\rho_Y) \cap \operatorname{Im}(r_1, \ldots, r_{n_1}))$$

$$= \max \{\dim \operatorname{Im}(\rho_Y) + \dim \operatorname{Im}(r_1, \ldots, r_{n_1}) - n_1(k+1),\ 0\}$$

$$= \max \{\dim \mathbf{L}_d(k^{n_1}) - \dim \mathbf{L}_d((k+1)^{n_1}) + n_1(\dim \mathbf{L}_k(m^{n_2}) - \dim \mathbf{L}_{k-1}(m^{n_2})) - n_1(k+1),\ 0\}.$$

We can now formulate the following claim.

Claim 3.2. The dimension of $\dim(\mathrm{Im}(\rho_Y) \cap \mathrm{Im}(r_1, \ldots, r_{n_1}))$ is equal to

$$\begin{cases} n_1(k+1) & \text{if } \dim \mathbf{L}_d((k+1)^{n_1}) \geq 0, \\ & \quad \dim \mathbf{L}_{k-1}(m^{n_2}) \geq 0; \\ n_1\left(\frac{k(k+3)}{2} + 1 - n_2\frac{m(m+1)}{2}\right) & \text{if } \dim \mathbf{L}_d((k+1)^{n_1}) \geq 0, \\ & \quad \dim \mathbf{L}_k(m^{n_2}) \geq 0,\ \dim \mathbf{L}_{k-1}(m^{n_2}) = 0; \\ \frac{d(d+3)}{2} + 1 - n_1\frac{k(k+1)}{2} & \text{if } \dim \mathbf{L}_d(k^{n_1}) \geq 0, \\ & \quad \dim \mathbf{L}_d((k+1)^{n_1}) = 0, \dim \mathbf{L}_{k-1}(m^{n_2}) \geq 0; \\ \frac{d(d+3)}{2} + 1 - n_1 n_2\frac{m(m+1)}{2} & \text{if } \dim \mathbf{L}_d((k+1)^{n_1}) = 0, \\ & \quad \dim \mathbf{L}_{k-1}(m^{n_2}) = 0, \\ & \quad \frac{d(d+3)}{2} + 1 - n_1 n_2 \frac{m(m+1)}{2} \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First consider the case $\dim \mathbf{L}_d((k+1)^{n_1}) \geq 0$ and $\dim \mathbf{L}_{k-1}(m^{n_2}) \geq 0$. Under this hypothesis,

$$\dim \mathbf{L}_d(k^{n_1}) - \dim \mathbf{L}_d((k+1)^{n_1}) + n_1(\dim \mathbf{L}_k(m^{n_2}) - \dim \mathbf{L}_{k-1}(m^{n_2})) - n_1(k+1)$$

is equal to

$$\frac{d(d+3)}{2} + 1 - n_1\frac{k(k+1)}{2} - \frac{d(d+3)}{2} - 1 + n_1\frac{k(k+3)}{2} + n_1 \\ - n_1 n_2 \frac{m(m+1)}{2} - n_1\frac{(k-1)(k+2)}{2} - n_1 + n_1 n_2\frac{m(m+1)}{2} - n_1(k+1).$$

All the terms containing d disappear and we are left with

$$\frac{n_1}{2}(-k^2 - k + k^2 + 3k + 2 + k^2 + 3k - k^2 - k + 2 - 2k - 2),$$

which is exactly $n_1(k+1)$. In particular it is nonnegative.

In the second case, $\dim \mathbf{L}_{k-1}(m^{n_2}) = 0$, the expression

$$\dim \mathbf{L}_d(k^{n_1}) - \dim \mathbf{L}_d((k+1)^{n_1}) + n_1 \dim \mathbf{L}_k(m^{n_2}) - n_1(k+1)$$

gives $-n_1\frac{k(k+1)}{2} + n_1\frac{(k+1)(k+2)}{2} + n_1\left(\frac{k(k+3)}{2} + 1 - n_2\frac{m(m+1)}{2}\right) - n_1(k+1)$, which is exactly equal to $n_1\left(\frac{k(k+3)}{2} + 1 - n2\frac{m(m+1)}{2}\right)$. By hypothesis, this quantity is nonnegative.

In the third case, we assume that $\dim \mathbf{L}_d(k^{n_1})$ and $\dim \mathbf{L}_{d-1}(m^{n_2})$ are both nonnegative and that $\dim \mathbf{L}_d((k+1)^{n_1}) = 0$. So the expression

$$\dim \mathbf{L}_d(k^{n_1}) - \dim \mathbf{L}_d((k+1)^{n_1}) + n_1(\dim \mathbf{L}_k(m^{n_2}) - \dim \mathbf{L}_{k-1}(m^{n_2})) - n_1(k+1)$$

under these hypotheses gives

$$\frac{d(d+3)}{2} + 1 - n_1\frac{k(k+1)}{2} + n_1\left(\frac{k(k+3)}{2} \right. \\ \left. + 1 - n_2\frac{m(m+1)}{2} - \frac{(k-1)(k+2)}{2} - 1 + n_2\frac{m(m+1)}{2}\right) - n_1(k+1),$$

which is equal to $\frac{d(d+3)}{2}+1+n_1\left(\frac{k(k+1)}{2}\right)$ as we wanted to show. By assumption this quantity is nonnegative, since it is the dimension of $\mathbf{L}_d(k^{n_1})$.

Finally, if we assume that

$$\dim \mathbf{L}_d((k+1)^{n_1}) = \dim \mathbf{L}_{k-1}(m^{n_2}) = 0,$$

then

$$\dim \mathbf{L}_d(k^{n_1}) - \dim \mathbf{L}_d((k+1)^{n_1}) + n_1(\dim \mathbf{L}_k(m^{n_2}) - \dim \mathbf{L}_{k-1}(m^{n_2})) - n_1(k+1)$$

is equal to

$$\frac{d(d+3)}{2}+1-n_1\frac{k(k+1)}{2}+n_1\frac{k(k+3)}{2}+n_1-n_1n_2\frac{m(m+1)}{2}-n_1(k+1).$$

We get precisely $\frac{d(d+3)}{2}+1-n_1n_2\frac{m(m+1)}{2}$. □

We then use the above computations to find $\dim \mathbf{L}_0$ in several cases, in particular applying $(*)$ and the two claims.

Claim 3.3. The above calculations imply

$$\dim \mathbf{L}_0 = \frac{d(d+3)}{2}+1-n_1n_2\frac{m(m+1)}{2}$$

if one of the following conditions holds:

i. $\frac{d(d+3)}{2}+1-n_1\frac{(k+1)(k+2)}{2}\geq 0$ and $\frac{(k-1)(k+2)}{2}+1-n_2\frac{m(m+1)}{2}\geq 0$;

ii. $\frac{d(d+3)}{2}+1-n_1\frac{(k+1)(k+2)}{2}\geq 0$, $\frac{(k-1)(k+2)}{2}+1-n_2\frac{m(m+1)}{2}\leq 0$ and $\frac{k(k+3)}{2}+1-n_2\frac{m(m+1)}{2}\geq 0$;

iii. $\frac{d(d+3)}{2}+1-n_1\frac{(k+1)(k+2)}{2}\leq 0$, $\frac{(k-1)(k+2)}{2}+1-n_2\frac{m(m+1)}{2}\geq 0$ and $\frac{d(d+3)}{2}+1-n_1\frac{k(k+1)}{2}\geq 0$;

iv. $\frac{d(d+3)}{2}+1-n_1\frac{(k+1)(k+2)}{2}\leq 0$, $\frac{(k-1)(k+2)}{2}+1-n_2\frac{m(m+1)}{2}\leq 0$ and $\frac{d(d+3)}{2}+1-n_1n_2\frac{m(m+1)}{2}\geq 0$.

In the same way, we notice that $\dim \mathbf{L}_0 = 0$ if all the inequalities

a) $\frac{d(d+3)}{2}+1-n_1\frac{(k+1)(k+2)}{2}\leq 0$,

b) $\frac{(k-1)(k+2)}{2}+1-n_2\frac{m(m+1)}{2}\leq 0$,

c) $\frac{d(d+3)}{2}+1-n_1n_2\frac{m(m+1)}{2}\leq 0$

hold.

Proof. Suppose $\dim \mathbf{L}_d((k+1)^{n_1})\geq 0$ and $\dim \mathbf{L}_{k-1}(m^{n_2})\geq 0$. Then $\dim\ker(\rho_Y)+\dim\ker(r_1,\dots,r_{n_1})+\dim(\mathrm{Im}(\rho_Y)\cap \mathrm{Im}(r_1,\dots,r_{n_1}))$ is equal to

$$\frac{d(d+3)}{2}+1-n_1n_2\frac{m(m+1)}{2}-n_1(k+1)+n_1(k+1)$$

which is $\frac{d(d+3)}{2}+1-n_1n_2\frac{m(m+1)}{2}$. We also note that, under the given assumptions, this quantity is nonnegative since it is given as a sum of two dimensions.

If $\dim \mathbf{L}_d((k+1)^{n_1}) \geq 0, \dim \mathbf{L}_k(m^{n_2}) \geq 0$ and $\dim \mathbf{L}_{k-1}(m^{n_2}) = 0$, then the sum

$$\frac{d(d+3)}{2} + 1 - n_1 \frac{(k+1)(k+2)}{2} + n_1 \left(\frac{k(k+3)}{2} + 1 - n_2 \frac{m(m+1)}{2} \right)$$

gives $\frac{d(d+3)}{2} + 1 - n_1 n_2 \frac{m(m+1)}{2}$ as stated.

In the third case, namely $\dim \mathbf{L}_d(k^{n_1}) \geq 0, \dim \mathbf{L}_{k-1}(m^{n_2}) \geq 0$ and $\dim \mathbf{L}_d((k+1)^{n_1}) = 0$, the sum $\dim\ker(\rho_Y) + \dim\ker(r_1, \ldots, r_{n_1}) + \dim(\operatorname{Im}(\rho_Y) \cap \operatorname{Im}(r_1, \ldots, r_{n_1}))$ is equal to

$$\frac{d(d+3)}{2} + 1 - n_1 \frac{k(k+1)}{2} + n_1 \left(\frac{(k-1)(k+2)}{2} + 1 - n_2 \frac{m(m+1)}{2} \right)$$

which again gives $\frac{d(d+3)}{2} + 1 - n_1 n_2 \frac{m(m+1)}{2}$.

If $\dim \mathbf{L}_d((k+1)^{n_1}) = \dim \mathbf{L}_{k-1}(m^{n_2}) = 0$ and also

$$\frac{d(d+3)}{2} + 1 - n_1 n_2 \frac{m(m+1)}{2} \geq 0,$$

then, since $\dim\ker(\rho_Y) + \dim\ker(r_1, \ldots, r_{n_1}) = 0$,

$$\dim\ker(\rho_Y) + \dim\ker(r_1, \ldots, r_{n_1}) + \dim(\operatorname{Im}(\rho_Y) \cap \operatorname{Im}(r_1, \ldots, r_{n_1}))$$

$$= \dim(\operatorname{Im}(\rho_Y) \cap \operatorname{Im}(r_1, \ldots, r_{n_1})) = \tfrac{d(d+3)}{2} + 1 - n_1 n_2 \tfrac{m(m+1)}{2}.$$

Finally, to prove the last statement, both $\dim\ker(\rho_Y) + \dim\ker(r_1, \ldots, r_{n_1})$ and $\dim(\operatorname{Im}(\rho_Y) \cap \operatorname{Im}(r_1, \ldots, r_{n_1}))$ vanish so their sum is zero as well, proving that $\dim \mathbf{L}_0 = 0$. □

4. Proof of Theorem 2

For the proof of Theorem 1 fix numbers d, m, n_1 and n_2.

First assume that the virtual dimension

$$\frac{d(d+3)}{2} + 1 - n_1 n_2 \frac{m(m+1)}{2}$$

of the vector space $\mathbf{L}_d(m^{n_1 n_2})$ is positive. We will show that it is equal to the dimension of $\mathbf{L}_0$. In Section 3 we proved that the equality holds, if we can choose k that satisfies one of the conditions i), ii), iii) or iv).

If $\frac{d(d+3)}{2} + 1 - n_1 \frac{(k+1)(k+2)}{2} \geq 0$, then the only condition we need to check is that $\frac{k(k+3)}{2} + 1 - n_2 \frac{m(m+1)}{2} \geq 0$. Then either i) or ii) will be satisfied.

In the other case $\frac{d(d+3)}{2} + 1 - n_1 \frac{(k+1)(k+2)}{2} \leq 0$, then, since we're assuming that $\frac{d(d+3)}{2} + 1 - n_1 n_2 \frac{m(m+1)}{2} \geq 0$, we just need to show that $\frac{d(d+3)}{2} + 1 - n_1 \frac{k(k+1)}{2} \geq 0$. Then either iii) or iv) will be true.

Define k to be the integer between

$$k_l = \frac{1}{2}\left(-3 + \sqrt{1 + 4n_2 m(m+1)}\right) \text{ and } k_u = \frac{1}{2}\left(-1 + \sqrt{1 + 4n_2 m(m+1)}\right).$$

Such an integer exists since $k_u = k_l + 1$. Then $\frac{k_l(k_l+3)}{2} + 1 - n_2 \frac{m(m+1)}{2} = 0$ together with $k_l \leq k$ implies

$$\frac{k(k+3)}{2} + 1 - n_2 \frac{m(m+1)}{2} \geq 0.$$

In the same way, $\frac{d(d+3)}{2}+1-n_1\frac{k_u(k_u+1)}{2}=\frac{d(d+3)}{2}+1-n_1n_2\frac{m(m+1)}{2}\geq 0$ together with $k\leq k_u$ implies

$$\frac{d(d+3)}{2}+1-n_1\frac{k(k+1)}{2}\geq 0.$$

This proves that the chosen k satisfies at least one set of the inequalities i), ii), iii) or iv).

Next assume that the virtual dimension

$$\frac{d(d+3)}{2}+1-n_1n_2\frac{m(m+1)}{2}$$

of the vector space $\mathbf{L}_d(m^{n_1n_2})$ is negative. We will show that in this case $\dim\mathbf{L}_0=0$. Recall from the previous section that $\dim\mathbf{L}_0=0$ if we can find an integer k such that all inequalities a), b), c) hold. As before, define k to be an integer between

$$k_l=\frac{1}{2}\left(-3+\sqrt{1+4n_2m(m+1)}\right)\text{ and }k_u=\frac{1}{2}\left(-1+\sqrt{1+4n_2m(m+1)}\right).$$

The last inequality c) is automatically fulfilled. From

$$\frac{d(d+3)}{2}+1-n_1\frac{(k_l+1)(k_l+2)}{2}=\frac{d(d+3)}{2}+1-n_1n_2\frac{m(m+1)}{2}\leq 0$$

together with $k_l\leq k$ we get

$$\frac{d(d+3)}{2}+1-n_1\frac{(k+1)(k+2)}{2}\leq 0.$$

Also, $\frac{(k_u-1)(k_u+2)}{2}+1-n_2\frac{m(m+1)}{2}=0$ together with $k\leq k_u$ implies

$$\frac{(k-1)(k+2)}{2}+1-n_2\frac{m(m+1)}{2}\leq 0.$$

Therefore we proved that we can always choose an integer k in the linear system $\mathbf{L}_0=\mathbf{L}_0(d,k,m,n_1n_2)$ such that

$$\dim\mathbf{L}_0=\max\left\{\frac{d(d+3)}{2}+1-n_1n_2\frac{m(m+1)}{2},\ 0\right\}.$$

This finishes the proof of Theorem 2.

5. Applications

5.1. **A proof of Evain's theorem.** It is well known that the system of plane curves of degree d passing through 4 general points with homogeneous multiplicity m is non-special. Proof of this fact can be found in [1]. Given this, Theorem 1 implies Evain's theorem [4]: namely that all systems of plane curves of degree d through 4^h points with homogeneous multiplicity m are non-special.

5.2. **The problem of 9^h points.** With similar methods, we can prove another theorem.

Theorem 3. *The linear systems $\mathcal{L}_d(m^{9^h})$ are non-special.*

As in the previous case, this theorem follows as a corollary from our main Theorem 1; since the case of 9 points is known and completely analogous to the case of 4 points.

5.3. **The problem of $4^h 9^k$ points.** The following theorem is also a consequence of our Theorem 1.

Theorem 4. *The linear systems $\mathcal{L}_d(m^{4^h 9^k})$ are non-special for all integers d, m, h and k.*

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